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Tolerance and sensitivity in the fuse network

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Abstract. We show that depending on the disorder, a small noise added to the threshold distribution of the fuse network may or may not completely change the subsequent breakdown process. When the threshold distribution has a lower cutoff at a finite value and a power-law dependence towards large thresholds with an exponent which is less than 0.16 ± 0.03, the network is not sensitive to the added noise, otherwise it is. The transition between sensitivity and insensitivity appears to be second order, and is related to a localization–delocalization transition observed earlier in such systems.

Suppose one manufactures a set of machine parts, say, parts which are identical to within some predetermined tolerance. One may ask whether this predetermined tolerance is small enough so that one may with reasonable certainty predict the strength and fracture properties of each member of the set. If it is, then testing one member of the set will give a representative idea of what is to be expected from the other members. However, fracture is a highly correlated process where singularities in the stress field are caused by fractures opening, and these singularities in turn produce more cracks. Thus, it is intuitively very likely that the eventual fractures forming will be very sensitive to what may appear as small initial differences between various samples, with a result, for example, that the fracture toughness may vary considerably from sample to sample. Is it therefore possible to define the concept of tolerance in the sense that if two members of a set are equal to within a certain limit, they will have the same fracture properties? It is the aim of this letter to discuss this question. We use the fuse model, originally introduced by de Arcangelis *et al* [1], as a model system. This model has proven itself to be extremely rich in addition to capturing some of the essential features of brittle fracture — see [2] for a throrough discussion of this. We find that whether rupture develops in a manner which is unpredictable in the sense discussed above does not depend on the noise distribution, but on the distribution of local strengths of the system itself. For some strength distributions, the network is sensitive to the initial added noise, and for other distributions it is not. We also find numerically that there is a second-order phase transition separating the sensitive from the non-sensitive regime. We suggest that this phase transition reflects a localization–delocalization transition previously seen in this system [3,4].

We work with a square lattice of size $L \times L$ oriented at 45° between two bus bars. In the direction parallel to the bus bars, the lattice is periodic. Each bond in the lattice is a fuse, i.e. it acts as an ohmic resistor as long as the current i it carries is lower than some threshold current t. If the current exceeds this threshold, the fuse 'blows' and turns irreversibly into

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an insulator. Each bond, i.e. fuse, is assigned a threshold t from a (cumulative) statistical distribution $P(t)$. There are no spatial correlations built into the way the threshold values are assigned. After the thresholds have been assigned, we imagine setting up a potential difference between the two bus bars, which is slowly increased. As fuse after fuse reaches its threshold current and burns out, the conductivity across the network decreases until it drops to zero. At this point a band of blown fuses has formed which cuts the network in two.

The breakdown process is highly complex, with long-range correlations developing as it advances towards rupture. This is most easily seen through the way we actually simulate the breakdown process numerically. Each time a fuse has blown, we recalculate the current distribution within the network by solving the Kirchoff equations for a unit voltage difference between the bus bars. For each bond k, we calculate the ratio i_k/t_k. We then search for the maximum of this ratio, $\max_k(i_k/t_k)$. The corresponding bond is the next to be cut, and this will happen at a voltage difference $1/\max_k(i_k/t_k)$. At the beginning of the breakdown process when few bonds have been cut, the current distribution is very narrow, i.e. the bonds all carry nearly the same current (and when no bonds have been cut they all carry exactly the same current). Thus, the bonds which are likely to break, i.e. those bonds whose ratio i_k/t_k is large, are those whose thresholds t_k are small; these are the weak bonds. However, as the rupture process evolves, the current distribution becomes wider and wider, and eventually a large ratio i_k/t_k may be caused by a large current i_k rather than a small threshold t_k. Towards the end of the breakdown process, this is the typical case. We may therefore split the breakdown process into three regimes [5]: (1) the disorder regime, where the bonds break because they are weak, so that it is the threshold distribution which governs the breakdown process, (2) the competition regime where the current distribution is roughly as wide as the threshold distribution, causing the breakdown to be a subtle cooperative process between the two distributions, and finally, (3) the current-governed regime where bonds break because they carry a large current. This regime manifests itself through a single macroscopic unstable crack eating its way through the network, and eventually breaking it apart. The disorder regime (1) is characterized by the nucleation of microcracks, and is essentially a process in which bonds are cut at random (since there are no spatial correlations in the way the thresholds were assigned). The competition regime (2) resembles superficially the disorder regime, but the long-range correlations that are developing through the current distribution result in subtle scaling laws, for example, in the current–voltage characteristics of the network [5].

The picture we have presented above is generic. Nothing has been said about the threshold distribution, $P(t)$. It has earlier been argued that in the limit of infinitely large lattices, the breakdown process is completely determined by the strength of the singularities of the threshold distribution for $t \to 0$ and $t \to \infty$ [3]. These two singularities, $P(t) \sim t^{\phi_0}$ for $t \to 0$ and $1 - P(t) \sim t^{-\phi_\infty}$ for $t \to \infty$, are characterized by the two exponents ϕ_0 and ϕ_∞. Depending on these two exponents—the control parameters—the fracture process develops differently, even though the general characteristics sketched above remain the same. The behaviour of the breakdown process may be classified into distinct *phases*. There are at least five such phases. (1) If either ϕ_0 or ϕ_∞ is zero, the disorder is so large that the current distribution is never able to compete with it. The breakdown process in this case remains a random percolation process until the lattice is broken apart. This is because the only constraint on the breakdown process from the current distribution is that the bond that may break carries a current different from zero. This leads to the breakdown process being a *screened percolation* process [6], and therefore, belonging to the universality class of standard percolation. In particular, a finite percentage of bonds must be broken in order to break the network apart in the limit of infinitely large lattices. (2) If both ϕ_0 and ϕ_∞ are small — a mean-field calculation [3] puts both the critical ϕ_0 and ϕ_∞ at the value two—the process is no longer in the universality class of percolation, but still a finite percentage of bonds must be broken in order to break the network apart. (3) When ϕ_0 is small (less than two according to mean-field theory) and ϕ_∞ is large (i.e. larger than two), the network is very weak in the sense that only a fraction of bonds approaching zero needs to be broken in order to break the network apart. More precisely, if the lattice size is $L \times L$, the number of bonds to break scales as $L^{1.7}$, irrespective of the two control parameters ϕ_0 and ϕ_∞. The fracture process proceeds by a significant number of 'microcracks'—small clusters of broken bonds—developing before the process goes unstable and a macroscopic crack eventually develops, breaking the network apart. (4) When ϕ_0 is large (larger than two according to mean-field theory) and ϕ_∞ is small (smaller than two), the number of bonds that breaks before the entire network ruptures scales with the lattice size L to a power smaller than two, but which, unlike the previous case, now depends on the two control parameters [7]. As with phase (3), the number of bonds belonging to the final unstable crack breaking the network apart forms a zero subset of the total number of bonds that break. The key difference between phases (3) and (4), in terms of the way the disorder influences the fracture process, is that in phase (3) microcracks are being induced because bonds are very weak, while in phase (4) new microcracks open as those already in existence are stopped by very strong bonds. (5) This is where both ϕ_0 and ϕ_∞ are large. According to a mean-field calculation, 'large' means here larger than two. An indirect numerical calculation [4] puts the transition for $\phi_0 = \infty$ at $\phi_\infty = 0.16$. Another numerical calculation [8] puts the critical ϕ_0 at 1.4 for $\phi_\infty \to \infty$. The defining characteristic of this phase is *localization*. That is, few microcracks form before one of them becomes unstable and eventually cuts the lattice apart. By 'few' we mean that the fraction of broken bonds that do not belong to the unstable crack which eventually breaks the network, goes to zero as $L \to \infty$.

In this letter we introduce the concept of *sensitivity* in connection with fracture, borrowing it from the study of cellular automata [9] where it is known as *damage spreading*, a name which would be very misleading in connection with fracture, as it has nothing to do with the already well established concept of damage in fracture. Let us define the concept operationally in terms of the fuse network. We set up two *identical* networks—identical in the sense that each corresponding fuse in the two lattices has the same threshold value assigned to it. The breakdown of these two copies will of course evolve identically. Let us now choose a bond in, say, lattice A and set its threshold value to infinity, thus making it unbreakable. The threshold of the corresponding fuse in lattice B is set to zero, thus making sure it will break immediately. In this way we introduce a small difference between the two copies, and the question we pose is: Will the difference between the two lattices, in terms of where the cracks appear, grow as the fracture process proceeds, or will stay small, that is, of the order of one bond. If the difference grows, the networks are sensitive.

The interest in defining such a concept lies in the concept of *tolerance* discussed in the introduction. In terms of the fuse network, we imagine producing a set of such networks, all with the same distribution of fuse strengths except for an added noise making each pair of corresponding fuses slightly different. The strength and distribution of this added noise correspond to the tolerance. If the added noise is sufficient to induce the initial microcracks appearing under load to happen at different places from copy to copy, and if they are sensitive in the sense introduced above, then the fracturing of different copies will develop differently. In other words, predictions on how other lattices will behave cannot be made from testing one single copy.

Thus, the question of whether an added noise in the assignment of thresholds is enough to change the breakdown properties of the network is a question of the noise being strong enough to change where the initial microcracks develop, and then if yes, whether the network is sensitive or not.

Whether the noise is sufficient to change the initial microcracks is a question of *order statistics* [10]. Let us assign the fuse strengths according to the rule

$$t_i = r_i^D \tag{1}$$

where r_i is a random number drawn from a uniform distribution between zero and one. D is the control parameter; small values of $|D|$ correspond to small disorder and large values of $|D|$, to large disorder. In terms of the cumulative threshold distribution, $P(t)$, which is the probability to find a threshold value smaller than or equal to t, this corresponds to

$$P(t) = \begin{cases} t^{1/D} & \text{where } 0 < t < 1 \text{ if } D > 0 \\ 1 - t^{1/D} & \text{where } 1 < t < \infty \text{ if } D < 0. \end{cases} \tag{2}$$

Thus, in terms of the two control parameters ϕ_0 and ϕ_∞, we see that when $D < 0$, $D = -1/\phi_\infty$ while $\phi_0 \to \infty$, but for $D > 0$, $D = 1/\phi_0$ while $\phi_\infty \to \infty$. We will base our arguments on this distribution, even though it does not cover all relevant disorders (for which both ϕ_0 and ϕ_∞ simultaneously are finite). However, it is easy, as we will show, to extrapolate our results to other regions of the parameter space. Let us also assume that the cumulative distribution of added noise is of the form

$$R(\delta t) = \left(\frac{\delta t}{\delta t_{\max}}\right)^\eta \tag{3}$$

where $0 < \delta t < \delta t_{\max}$ and $\eta > 0$.

Let us now assume that $D > 0$. Then the threshold distribution for the bonds *including* the noise is

$$P_R(t) = \int_0^t \mathrm{d}\tau \int_0^1 \mathrm{d}u \int_0^{\delta t_{\max}} \mathrm{d}v\, p(u) r(v) \delta\left(\tau - (u + v)\right) \tag{4}$$

where $p(t) = \mathrm{d}P(t)/\mathrm{d}t$ and $r(t) = \mathrm{d}R(t)/\mathrm{d}t$. Integrating out the Dirac delta function gives

$$P_R(t) = \frac{\eta}{D(\delta t_{\max})^\eta} \int_0^t \mathrm{d}u \int_{\max(u-\delta t_{\max},0)}^{\min(u,1)} \mathrm{d}v\, v^{1/D-1}(u-v)^{\eta-1}\,. \tag{5}$$

For t of the order of $\delta t_{\max}$ or smaller, the distribution $P_R(t)$ behaves as

$$P_R(t) = a t^{1/D+\eta} \tag{6}$$

where a is a positive constant. For larger t it behaves as

$$P_R(t) = P(t) = t^{1/D}\,. \tag{7}$$

Suppose we draw N $(= 2 \times L^2)$ thresholds from the distribution $P(t)$. We order them so that $t_{(1)} \leqslant t_{(2)} \leqslant \ldots \leqslant t_{(N)}$. The expectation value for the threshold number k in this sequence is

$$t_{(k)} = \left(\frac{k}{N+1}\right)^D \tag{8}$$

where we have used the general expression $P(t_{(k)}) = k/(N+1)$. We also form an ordered sequence of the thresholds obtained with perturbed distribution (5), $t'_{(1)} \leqslant t'_{(2)} \leqslant \ldots \leqslant t'_{(N)}$. For small values of k, the expectation value of the kth element of this sequence is

$$t'_{(k)} = \left(\frac{k}{a(N+1)}\right)^{D/(1+D\eta)}. \tag{9}$$

We may now pose the question of whether the added noise changes the sequence of weak bonds or not? If the sequence is changed, we have

$$t'_{(k)} > t_{(k+1)} \,. \tag{10}$$

Using equations (8) and (9) in this inequality leads to the expression

$$1 > a^{1/D\eta} \left(1+\frac{1}{k}\right)^{1+1/D\eta} \left(\frac{k}{N+1}\right). \tag{11}$$

In particular, for large N *and* k, (11) may be written as

$$k < \left(\frac{1}{a}\right)^{1/D\eta} N. \tag{12}$$

For any fixed k, (11) and (12) are always true for large enough N. Thus, no matter how small the added noise is, it *does* change the sequence of the weakest bonds. It should be noted in this argument that the upper cutoff in the noise distribution, $\delta t_{\max}$, does not enter in the discussion: No matter how small it is, the noise will be relevant for the weakest bonds when the network is large enough.

We now repeat this analysis for $D < 0$. The noise distribution is still given by equation (3), while the threshold distribution now is

$$P(t) = 1 - t^{1/D} \tag{13}$$

for $1 < t < \infty$. The threshold distribution after adding the noise is

$$P_R(t) = \frac{\eta}{D(\delta t_{\max})^\eta} \int_1^t \mathrm{d}u \int_{\max(u-\delta t_{\max},1)}^{u} \mathrm{d}v \; v^{1/D-1}(u-v)^{\eta-1} \,. \tag{14}$$

For t close to 1, we have

$$P_R(t) = \frac{a}{D}(t-1)^{1+\eta} \tag{15}$$

rather than

$$P(t) = \frac{1}{D}\,(t-1) \tag{16}$$

for the unperturbed threshold distribution. Again ordering the sequence of thresholds from the unperturbed distribution and the perturbed distribution, equation (10) leads to the inequality

$$1 > (D^{D\eta}a)^{1/D\eta} \left(1+\frac{1}{k}\right)^{1+1/D\eta} \left(\frac{k}{N+1}\right) \tag{17}$$

which is satisfied for sufficiently large networks. Thus, also in this case, the noise will change the sequence of the fuses having the smallest thresholds. As before, the upper cutoff of the added noise, δt_{max} does not enter the discussion.

Hence, this chain of arguments leads to the conclusion that whether or not the fuse network is sensitive to the added noise, does not depend on the noise for large enough systems. The next question is whether it depends on the threshold distribution, $P(t)$ as such. Thus we investigate whether the system is *sensitive* or not in the sense introduced above: starting with two identical copies of a fuse network except for one pair of fuses, which are made infinitely weak and infinitely tough, respectively, we measure whether the two copies evolve differently or identically during breakdown.

It should be noted here that if we find that the network is sensitive with respect to changing the threshold of only one bond, then it is sensitive with respect to adding everywhere a noise to the threshold distribution. However, the opposite is not true: as we will see for a certain regime of disorder ($D > 0$), the network is *not* sensitive with respect to changing the threshold value of a single bond, but is sensitive with respect to adding noise everywhere.

We have simulated the fuse network numerically, generating ensembles containing from 1000 to 200 samples each and ranging in size from 10×10 to 128×128, using the threshold distribution (1) with $-3 < D < 1$. Each time a fuse blows, we recalculate the current distribution in what is left of the network, using the conjugate gradient method [11]. This algorithm is eminently parallelizable, and ran very efficiently on a Connection Machine CM5 computer. Each sample consists of two copies of the same network, but with one central bond different. Both networks are completely broken apart, and afterwards the macroscopic crack breaking each of the two lattices is identified and compared. The order parameter we have used is

$$S = \frac{\frac{1}{2}\sum_{i,j} \mathrm{xor}\left(n_i^A, n_j^B\right)}{\max\ \left(\sum_i n_i^A, \sum_j n_j^B\right)} \tag{18}$$

where n_i^A is unity if bond i of lattice A belongs to the final crack, otherwise it is zero. Likewise, n_j^B refers to lattice B. The (logical) function xor is one if the two arguments are different, otherwise it is zero, i.e. $\mathrm{xor}\,(1,0) = \mathrm{xor}\,(0,1) = 1$ and $\mathrm{xor}\,(1,1) = \mathrm{xor}\,(0,0) = 0$. If the system is sensitive, we find that $S \to 1$, and if it is not, $S \to 0$. In figure 1, we plot S as a function of the control parameter D defined in equation (1) for various lattice sizes. As is evident, there is a first-order transition (i.e. a discontinuity) in the order parameter S for $D = 0$, and a second-order transition (i.e. the slope of $S = S(D)$ diverges) for a negative $D = D_c$. We determine $D_c = -6.2 \pm 1.0$ by plotting $D_{\mathrm{eff}}(L)$ as a function of $L^{-1/\nu}$, where $D_{\mathrm{eff}}(L)$ is the D-value for lattice size L where $S(D)$ has the largest slope, and $1/\nu$ is chosen so that $D_{\mathrm{eff}}(L)$ falls on a straight line. We show our fit in figure 2 and the exponent $1/\nu$ determined from here gives an estimate of the correlation length exponent $\nu = 5 \pm 2$.

Thus, we see that there is a window $-6.2 < D < 0$ in which the fuse network is sensitive. If $D > 0$, then $\phi_0 = 1/D$ and $\phi_\infty \to \infty$. Within this range of ϕ_0-values the network undergoes a localization–delocalization transition, which numerical simulations [8] put at $\phi_0 = 1/D = 1.4$. *There is no trace of this transition in the order parameter S.* For negative D, there is a sensitive phase, which exists for $\phi_\infty > 1/6.2 = 0.16$ and $\phi_0 \to \infty$. The localization–delocalization transition in this range of parameters has been, as already pointed out, numerically determined [4] to appear at $\phi_\infty = -1/D = 0.16$. The phase transition in S seen at $D = -6.2$ is, therefore, likely to be related to this transition.

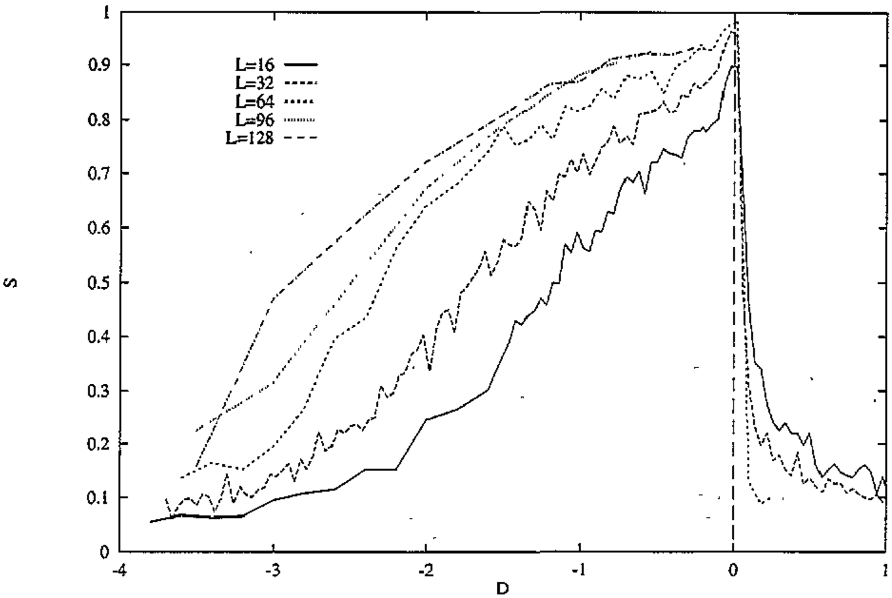

Figure 1. The order parameter S defined in equation (18), as a function of the control parameter D defined in equation (1) for lattice sizes 16 (200 samples), 32 (200), 64 (200), and 128 (200). The difference between the two copies constituting each sample is the strength of a single bond.

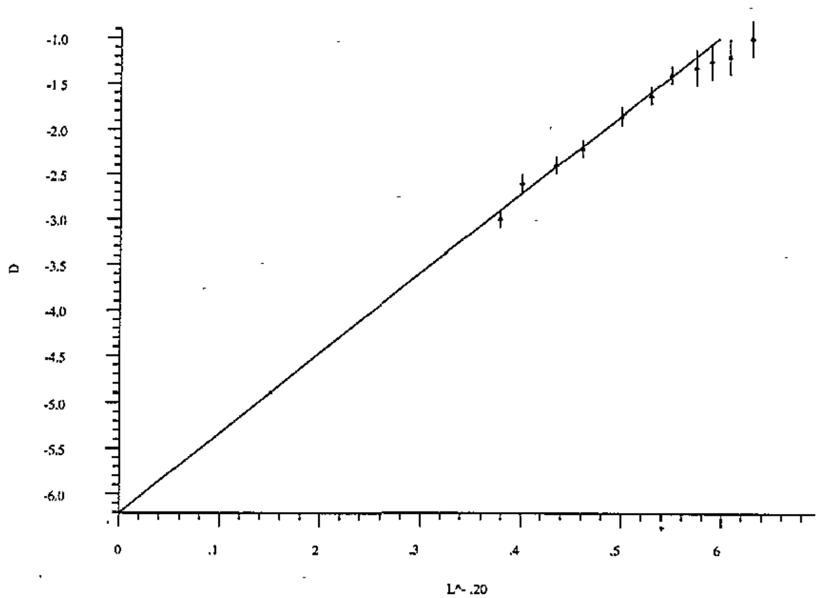

Figure 2. D_{eff} determined from figure 1 and additional lattice sizes not shown, plotted against $L^{-1/\nu}$. From this plot we estimate that $D_c = -6.2 \pm 1.0$ and $\nu = 5 \pm 2$.

Why does there seem to be a connection between the localization–delocalization transition and the existence of a sensitivity–insensitivity transition for $D < 0$, but not for $D > 0$? If, in the localized phase, the first bond to break initiates the final macrocrack, we expect the localized phase to coincide with the sensitive phase. This seems to happen for $D < 0$. We may understand this by noting that in this case the localization–delocalization transition is caused by crack arrest: The microcracks are *a priori* all unstable. However, if the distribution of strong elements is sufficiently large, the crack is stopped by hitting a bond strong enough for the enhanced currents around the crack tip to be insufficient to continue the growth of this particular crack. Thus, when the disorder is small enough, the first microcrack cannot be 'held' back, and both sensitivity and localization follows. On the other hand, when the system is in the delocalized regime, a diverging (with the lattice size) number of microcracks develop before one of them eventually goes unstable and develops into a macroscopic crack. Thus, a small initial perturbation among the microcracks will typically not affect the macrocrack that eventually develops, and the network is thus not sensitive.

For $D > 0$, the localized phase is different. In this case we have a localized phase even though there is a diverging number (with lattice size) of microcracks forming before one of them goes unstable and grows into the macroscopic crack that breaks the network apart. This is possible since the ratio between the number of bonds forming the microcracks and the number of bonds belonging to the final crack goes to zero: the total number of bonds that has broken throughout the entire fracture process is dominated by the final crack. In this case, the probability that the one artificially induced microcrack in one of the two copies actually should be the one that goes unstable falls to zero with the lattice size. This happens since there is a power-law distribution of bonds whose thresholds are very weak, so that there always is a 'mist' of microcracks before one goes unstable, no matter how fast this power law distribution falls off if the lattice is large enough. Thus, there will be no sensitive phase in this case, even though there is a localized phase.

We now return to the question of sensitivity in connection with the added noise in the threshold distribution. In figure 3, we show S as a function of D for an added disorder drawn from a uniform distribution between zero and $\delta t_{\max} = 0.1$. There is the same second-order transition at $D = -6.2 \pm 1.0$ in this case as there is for the case when the difference between the two copies is limited to one pair of bonds. This is no surprise from the above discussion. However, for $D > 0$, there *is* a difference: now, there is a sensitive phase for all $D > 0$, while there was none when only a single bond was changed. We interpret this in the following way. As the added noise affects the ordering of all weak bonds, and not only a single one, we expect that the one eventually leading to the final crack is also affected.

We conclude by recapitulating what has been found. We have investigated whether the fuse model is sensitive to the addition of noise in the threshold distribution, i.e. whether two networks, identical except for the added noise, develop the same macroscopic cracks or not. The disorder in the fuse model is completely described by two parameters. We have investigated the sensitivity of the model along a curve in this two-dimensional parameter space by two very different types of noise. When a single bond is made unbreakable in one copy and extremely weak in the other, we get a sensitive region for $-6.2 < D < 0$. At the lower end the order parameter disappears continuously with a large slope, while on the other side the order parameter jumps discontinuously to zero. When a weak noise is added everywhere, the sharp jump at $D = 0$ from a sensitive to an insensitive region disappears and a sensitive region develops for $D > 0$. The negative D region remains unchanged. We identify this transition with a localization–delocalization transition. The disappearance of the second insensitive region when going from strong local disorder to weak nonlocal

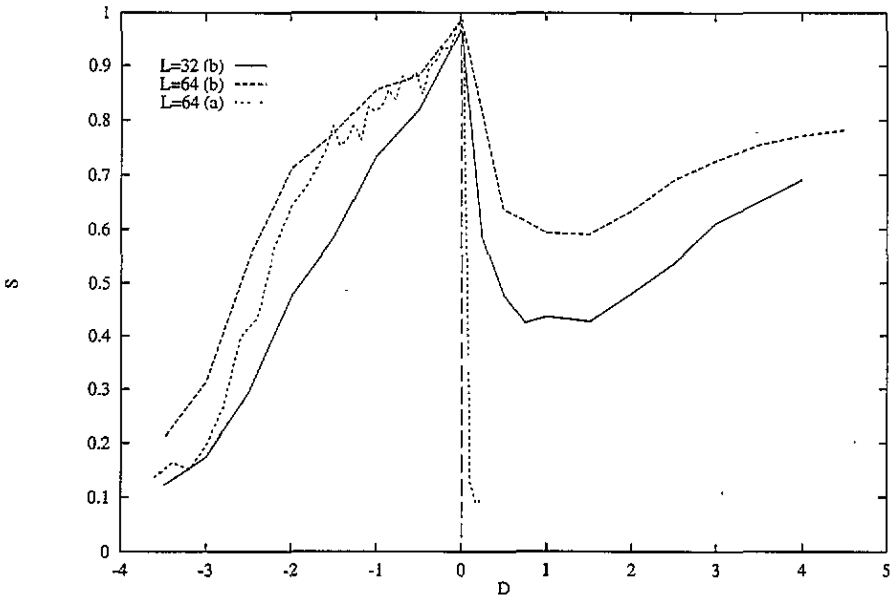

Figure 3. The order parameter S for 200 samples of sizes 32×32 and 64×64, where the difference between the two copies in each sample is a noise drawn from a flat distribution between zero and 0.1 added to each bond (b). We show for comparison the corresponding curve for lattices of size 64×64 when the difference between each copy is a single bond ((a), as in figure 1).

disorder is due to differences in crack arrest mechanisms in the two parts of the parameter space.

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